Solution Q1.1:

$p_t = E(s_t|y_{t-1}) = P(s_t = 1|y_{t-1})$.

Set $p_y = p_t$ with $y_{t-1} = y$. $E\left(1 + y_t^2|y_{t-1} = y\right) = 1 + \omega + [p_y\alpha + (1 - p_y)]\, y^2$ So as $p_y \to 1$ when $y^2 \to \infty$, we find that we need $\alpha < 1$ for drift criterion with $d(y) = 1 + y^2$.

Thus nonlinear ARCH: we have "no-variance regime" with say $\tilde{\alpha} = 1$ and one "finite variance" regime with $\alpha < 1$ - but the process has finite variance provided $\alpha < 1$.

Solution Q1.2: By definition of (conditional) mixture models.

Solution Q1.3: $p_t^* = \frac{p_t f_{1,t}}{p_t f_{1,t} + (1-p_t) f_{0,t}}$ and is the conditional smoothed probability $p_t^* = P(s_t = 1|y_t, y_{t-1})$.

Algorithm step n: compute p_t^* for $\hat{\theta}_{n-1}$. Next, solve numerically with p_t^* fixed, $\sum_{t=1}^T \left(\frac{y_t^2}{\sigma_{1,t}^2} - 1\right) p_t^* \frac{y_t^2}{\sigma_{1,t}^2} = 0$ and update $\hat{\theta}_n$.

Solution Q1.4: Given p_t one could set $s_t = 1$ if $u_t \in [0, p_t]$ and $s_t = 0$ otherwise, where u_t is $U[0,1]$. An example could be, the logistic, $p_t = \left(1 + \exp\left(-y_t^2\right)\right)^{-1}$ Misspecified IGARCH(1,1), $\hat{\alpha} + \hat{\beta} = 1$ and $\hat{\alpha} \approx 0$.

Q2.1: long memory - $\Delta^d v_t = \varepsilon_t$.

Q2.2: As s_t is iid and stationary for all $p_i \in]0,1[$ - then so is ε_t. (any form of the interval [0,1] will do).

$E\varepsilon_t^2 = E\sigma_{s_t}^2 = \sigma_1^2 p_1 + \sigma_2^2 p_2 := \omega$. $E\varepsilon_t\varepsilon_{t-1} = 0$ as z_t are iid mean zero.

$Ev_t^{*2} = \sum (1+\pi)^{2i}\omega = \frac{\omega}{1-(1+\pi)^2} < \infty$. v_t^* is a stationary solution to the equation for v_t.

Q2.3: Use the EM-algoritm with:

$$L_{\text{EM}}(Y_T;\theta) = c + \sum_{i,j=1}^{2} \log p_{ij} \sum_{t=2}^{T} p_t^*(i,j) + \sum_{j=1}^{2}\sum_{t=1}^{T} p_t^*(j) f_\theta(v_t|v_{t-1}, j), \qquad (1)$$

where the important difference with the standard case considered in the notes is the term $f_\theta(v_t|v_{t-1}, j)$, where,

$$f_\theta(y_t|y_{t-1}, j) = \frac{1}{\sqrt{2\pi\sigma_j^2}} \exp\left(-\frac{1}{2\sigma_j^2}\left(\Delta v_t - \pi v_{t-1}\right)^2\right) \qquad (2)$$

and the smoothed probabilities are given by:

$$p_t^*(i,j) = P_{\hat{\theta}}(s_{t-1} = i, s_t = j \mid Y_T) \quad \text{and} \quad p_t^*(j) = P_{\hat{\theta}}(s_t = j \mid Y_T).$$

Q2.4: misspecified model (LM tests indicate non-normal residuals with ARCH effects) - the autoregressive root is $1 + \hat{\pi} = 0.84$ which may indicate non-stationarity. The $\hat{P}$ matrix seems nice and "stationary" - but really we cannot conclude much as the misspecification tests so clearly indicate wrong model.

Q2.5: Use Ito's: $f(u,V) = \exp(-bu)V$; $f_V = \exp(bu)$; $f_{VV} = 0$; $f_u = -b\exp(-bu)V$;

$$
\begin{aligned}
&d(\exp(bu)V_u) && (3)\\
&= \exp(-bu)(bV_u du + cdW_u) - (b\exp(-bu)V_u)du && (4)\\
&= \exp(-bu)adu + \exp(-bu)cdW_u && (5)
\end{aligned}
$$

Hence,

$$\exp(-bu)V_u = V_0 + \int_0^u \exp(-bs)cdW_s \quad (6)$$

$$V_u = \exp(bu)V_0 + \int_0^u \exp(b(u-s))cdW_s \quad (7)$$

We conclude:

$$v_n = \exp(b)v_{n-1} + \varepsilon_n \quad (8)$$

and hence, $\pi = \exp(b) - 1$. Or $b = \log(1+\pi)$ st $\hat{b} = \log(1-0.16) = \log(0.84) = -0.17$.

Stationary: $b < 0$ and also DF test for $\pi = 0 = b$.